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Anomalies, Sphalerons and Baryon Number Violation in Electro-weak Theory

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Abstract: In this series of three lectures, baryon number violation at high temperatures in the Weinberg-Salam model is discussed. The first lecture presents a discussion of anomalies, and how this is related to level crossing of energy levels in the Dirac equation for fermions in an external field. The second lecture discusses topological aspects of the Weinberg-Salam theory, and some related two dimensional models. The sphaleron solution of these theories is constructed. In the final lecture, the sphaleron is related to transition rates at finite temperature. In a simple quantum mechanics model, it is shown that sphalerons, not instantons, are responsible for transitions at high temperature. The sphaleron induced rate is then discussed in a solvable 1+1 dimensional model, which has many similarities to the Weinberg-Salam model. Finally, the result for the Weinberg-Salam model is derived, and is shown to be large for temperatures $T \geq 1\ Tev$.

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1 Introduction

It has been long believed that baryon number conservation at a non-negligable rate, if it exists at all, is a property of grand unified field theories, and should only be relevant at energy scales $E \geq 10^{15}\ GeV$. It is a remarkable well known fact, nevertheless, that in the electroweak theory, baryon number is non-conserved as a result of the U(1) anomaly.[1]–[3] The rate for this process was estimated by t'Hooft at zero temperature,[4] using electro-weak instantons.[5]–[7] We shall discuss this result later in these lectures. The basic conclusion from his estimate is that the rate is negligibly small. In terms of the electroweak coupling $\alpha' = g'^2/4\pi \sim 1/40$, the rate is

$$\Gamma/V \sim e^{4\pi/\alpha'} \sim 10^{-173} \tag{1}$$

To get the dimensions correct for this rate per unit volume, we must multiply in some scale typical of electroweak theory, say the Z boson Compton wavelength $\sim 10^{-32}$ *lightyears*. In any case, even after multiplying by a small space-time volume such as $10^{-128} year/lightyear^3$, the rate is so small that it is improbable that in the entire space-time volume of the universe, one baryon has decayed due to electro-weak processes. If we were to naively extrapolate this rate from instantons to high temperature, the only change we might expect is that the coupling constant would run with temperature, and therefore the rate never becomes significant in cosmology.[8]–[9]

There has been a revived interest in baryon decay in recent years due to the realization that there may be other processes which can cause baryon number change.[10]–[12] At high temperatures, a process which is called sphaleron induced decay exists.[12]–[13] It is a unique consequence of being in the high temperature limit of the theory, and in fact the process is non-existent at zero temperature. The computation of the rate for sphaleron induced processes is still controversial, since it seems to contradict naive instanton based arguments.[14] In these lectures, I will try to argue that these computations are reliable, and that the rate for baryon number violation in cosmology becomes larger than the expansion rate of the universe for temperatures $T \geq 1\ Tev$

To understand that there may be new processes which induce transitions at finite temperature which do not have a counterpart at zero temperature, we consider the simple example of a pendulum at finite temperature.[14] This pendulum is shown in Fig. 1, and the periodic nature of the pendulum potential is shown in Fig. 2. The potential is periodic in the angular variable θ which gives the angular displacement of the pendulum from its rest position. Notice that the pendulum may classically exhibit small oscillations around any of the local minimum, minima which are displaced by integer multiples of 2π, corresponding to multiple windings of the pendulum. The pendulum has a topological charge, which is the number of times the pendulum has wound around 2π. If we define the topological charge in Euclidian space, $t \rightarrow -it$, the charge may be identified as

$$Q = \int_0^\beta dt \; \dot{x} \tag{2}$$

where $\beta = 1/T$.

At finite temperature, with temperature large compared to the potential energy of the pendulum, it is clear that the pendulum winds around easily. At such a high temperature, the ensemble of states for the pendulum includes many states with energy larger than that of the potential energy. For these states, the pendulum continually winds, and θ samples all possible values.

We shall study the pendulum in more detail in a later lecture. Here we shall only note that if we tried to estimate the instanton contribution to the Euclidian action at finite temperature, we would get a large contribution corresponding to a great suppression of tunneling at high temperature. This instanton amplitude is ordinarily interpreted as the amplitude for the system to make a transition which winds the angular coordinate of the pendulum by a multiple of 2π. Recall that the amplitude of tunneling at high temperature (a result we shall derive in a later lecture) is

$$A \sim e^{-S_E} \tag{3}$$

where the Euclidian action is

$$S_E = \int_0^\beta dt \; \frac{1}{2}\dot{x}^2 + V(x) \tag{4}$$

Here $\beta = 1/T$ where T is the temperature, and the potential energy of the pendulum is $V(x)$. At high temperature, the potential can be ignored compared to the kinetic energy contribution. The instanton solution which changes topological charge is the solution to the equations of motion which in the Euclidian time β takes the pendulum from 0 to 2π. This solution is

$$x(t) = 2\pi t/\beta \tag{5}$$

The action for this solution is

$$S_{inst} = 2\pi^2 T \tag{6}$$

and diverges as $T \rightarrow \infty$. The instanton induced rate therefore exponentially approaches zero at high T.

The reason why the Euclidian action and formulation of field theory is important at finite temperature can be understood very simply. The ordinary path integral is an expression for the time evolution operator

$$lim_{t\rightarrow\infty} \langle out \mid e^{itH} \mid in\rangle \tag{7}$$

At finite temperature, we are on the other hand interested in

$$Tr\ e^{-\beta H} \tag{8}$$

When this expression is converted to a path integral, we see therefore that the length in time is finite, $t = \beta = 1/T$, and it is Euclidian $t \rightarrow -it$. The trace condition is satisfied if for boson fields $\phi(0,\vec{r}) = \phi(\beta,\vec{r})$. For fermion fields, the situation is more subtle, but it can be shown that $\Psi(0,\vec{r}) = -\Psi(\beta,\vec{r})$. The relevant action is the Euclidian action in a four volume (β, V), with fields satisfying these boundary conditions.

The outline of these lectures is as follows:

In the first lecture, I discuss in some detail U(1) anomalies and their physical interpretation. I derive the U(1) anomaly following closely the method of Fujikawa, which employs path integral techniques. I derive the result for 2 and 4 dimensional gauge theories. I then proceed to a heuristic discussion of the anomaly and level crossings of fermions in the presence of external fields. This allows for a physical interpretation of the anomaly as particle creation in these external fields. I then provide a more systematic and rigorous derivation of this result.

In the second lecture, following Manton, I use general topological arguments to show that there should exist a deformation of fields which connects field configurations of different topological charge in various field theory examples. I then argue that there should exist an unstable solution which yields the local energy density at the saddle point of this field deformation. This unstable solution, called the sphaleron, is explicitly constructed for both the O(3) sigma model in 1+1 dimensions and for the electro-weak theory in 3+1 dimensions. I explicitly show that the electro-weak sphaleron has topological charge of $\frac{1}{2}$.

In the third lecture, I show that the sphaleron is relevant for the computation of the transition rate between field configurations of different topological charge, and in the electro-weak theory provides a mechanism for changing baryon number. Following the analysis of Affleck[32], I derive expressions for the transition rate at finite temperature. These expressions are then used to compute the rate for the electroweak theory in a range of temperatures where weak coupling methods are valid. Finally, following the classic analysis of Mottola and Wipf,[20] I use the O(3) sigma model in 1+1 dimensions to argue that the rate should be unsuppressed even at high temperature where the weak coupling methods fail. The rate is estimated in the electro-weak theory when weak coupling techniques are no longer reliable using intuition gained from study of this model. Finally, I discuss the apparent contradiction between instanton estimates of transtions rates and sphaleron estimates, and their possible resolution.

2 Lecture 1: U(1) Anomalies, and Fermion Level Crossing

2.1 Fujikawa's Derivation of the U(1) Anomaly

In this first section, we shall derive the U(1) axial anomaly using the technique of Fujikawa.[15] As a result of this derivation, we shall show that this also implies a non-conservation of baryon number in electro-weak theory.

To derive this anomaly, we consider a single species of massive fermion interacting with a gauged vector potential. (I shall first consider a left-right symmetric theory, and then later generalize to the case where there is a left-right asymmetry as is the case for the Weinberg-Salam theory) The generalization to multiple numbers of fermions, and theories with scalars is straightforward, and we shall not carry this out. We shall also assume the theory has been continued to Euclidian space so that the path integral,

$$\int [dA][d\overline{\Psi}][d\Psi]\ e^{-S[\overline{\Psi},\Psi,A]} \tag{9}$$

is well defined. This analytic continuation takes $t \rightarrow -it$, $A^0 \rightarrow -iA^0$, and $\gamma^0 \rightarrow -i\gamma^0$. The path integral integration measure for the vector field is

$$[dA] = \Pi_{x,\mu,\alpha}\ dA^\mu_\alpha(x) \tag{10}$$

The integration measure for the fermions is Grassman. The action is

$$S = \int d^dx\ \overline{\Psi}[-i\gamma \cdot D + m]\Psi + \frac{1}{4}F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha \tag{11}$$

Here

$$D = \partial - ig\tau \cdot A \tag{12}$$

and

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + gf^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu \tag{13}$$

To derive the anomaly, we use that the path integral should be invariant under a change of integration variable. We make a local change of variable corresponding to a local U(1) chiral rotation:

$$\Psi(x) \rightarrow e^{i\gamma_5\alpha(x)}\Psi(x) \tag{14}$$

and

$$\overline{\Psi}(x) \rightarrow \overline{\Psi}(x)e^{i\gamma_5\alpha(x)} \tag{15}$$

This induces two changes in the path integral. The first is the change in the action. The second is a change induced in the integration measure. It is these two separate changes which must compensate one another. The change in the action is:

$$\delta S = (\partial_\mu\alpha(x))\overline{\Psi}\gamma^\mu\gamma_5\Psi + 2im\alpha(x)\overline{\Psi}\gamma_5\Psi \tag{16}$$

To understand the effect of a change in the functional measure, we need to know some properties of integration over Grassman variables. The first thing we need to know is that the only nontrivial integral for an N-dimensional Grassman space is

$$\int da_1 \cdots da_N\ a_1 \cdots a_N \tag{17}$$

Now under a change of variables,

$$a_i' = \sum_j C_{ij} a_j \tag{18}$$

we have

$$a_1' \cdots a_N' = det(C)\ a_1 \cdots a_N \tag{19}$$

since by the Grassmann algebra $\{a_i, a_j\} = 0$, which also implies $a_i^2 = 0$ In order to make this integral invariant under the coordinate parameterization, it must be true that

$$\Pi_i\ da_i' = det(C)^{-1} \Pi_i\ da_i \tag{20}$$

Under the infinitesimal chiral transformation above, the matrix C is

$$C = \{I + i\alpha(x)\gamma_5\}\ \delta(x-y) \tag{21}$$

Using

$$ln\ DetC = Tr\ lnC \tag{22}$$

we finally have to evaluate

$$det(C)^{-1} = exp\{\ Tr\ ln(1 + i\alpha\gamma_5)^{-1}\} \tag{23}$$

In this expression, the trace is over the coordinate and spatial indices of the operator C.

In general, the computation of this trace is fraught with ultra-violet divergences, and is ill-defined. To properly compute it, we must regulate in a gauge invariant way. This is most easily done by evaluating the trace in a basis which uses eigenstates of the operator

$$\gamma \cdot D\Phi_n = \lambda_n \Phi_n \tag{24}$$

We must evaluate for infinitesimal α

$$det(C)^{-1} = exp\{-i(\int dx\alpha(x) \sum_k \Phi_k^\dagger(x)\gamma_5\Phi_k(x))\} \tag{25}$$

The evaluation of the sum over k is singular for large k. We regulate it with a cutoff M,

$$\sum_k \Phi_k^\dagger(x)\gamma_5\Phi_k(x) \quad = \quad lim_{M\to\infty} \sum_k \Phi_k^\dagger(x) exp-[(\lambda_k/M)^2]\gamma_5\Phi_k(x) \tag{26}$$

$$= \quad tr\ \gamma_5 exp[-(\gamma\cdot D/M)^2]\delta(x-y) \tag{27}$$

$$= \quad lim_{M\to\infty, y\to x} \int \frac{d^dk}{(2\pi)^d}\ e^{ik\cdot(x-y)} \tag{28}$$

$$tr\ \gamma_5 exp(-[D^\mu D_\mu + \frac{1}{4}g^2[\gamma^\mu,\gamma^\nu]\tau\cdot F_{\mu\nu}]/M^2) \tag{29}$$

$$\tag{30}$$

In two dimensions, the gamma matrices are the 2×2 sigma matrices which we can choose as $\gamma_0 = \sigma_1$, $\gamma_1 = \sigma_2$, and $\gamma_5 = \sigma_3$ The first non-vanishing term from expanding the trace arises therefore from

$$\sum_k \Phi_k^\dagger(x)\gamma_5\Phi_k(x) \quad = \quad lim_{M\to\infty}\ \frac{1}{4}\ g^2\ tr\ \gamma_5[\gamma^\mu,\gamma^\nu]\tau\cdot F_{\mu,\nu} \tag{31}$$

$$\frac{g^2}{M^2}\int \frac{d^2k}{(2\pi)^2} exp(-k^2/M^2) \tag{32}$$

$$= \quad -\frac{g^2}{4\pi}\ tr\ \tau\cdot\epsilon^{\mu\nu}F_{\mu\nu} \tag{33}$$

In two dimensions, we see that we only get a contribution to the U(1) anomaly from the U(1) part of the gauge group. For example in an SU(2) gauge group, the trace over the group indices vanishes. Therefore for abelian gauge theory in 1+1 dimensions, we have

$$\sum_k \Phi_k^\dagger(x)\gamma_5\Phi_k(x) = -\frac{g^2}{4\pi}\ \epsilon\cdot F \tag{34}$$

In the four dimensional gauge theory, we have to expand to second order in the $F^{\mu\nu}$ term to get a non-vanishing result:

$$\sum_k \Phi_k^\dagger(x)\gamma_5\Phi_k(x) \quad = \quad lim_{M\to\infty}\ \frac{g^2}{16}\ tr\ [\gamma_5([\gamma_\mu,\gamma_\nu]F_{\mu\nu})^2] \tag{35}$$

$$\frac{1}{2M^4}\int \frac{d^4k}{(2\pi)^4} exp(-k^2/M^2) \tag{36}$$

$$= \quad -\frac{g^2}{(32\pi^2)}\ F^{\alpha d}_{\mu\nu}F^{\mu\nu}_{\alpha} \tag{37}$$

Here F^d denotes the dual of F,

$$F^d_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma} \tag{38}$$

We may now require that the change in the measure factor of the integration cancels the explicit change induced by the chiral rotation. (There are two identical factors from the integration measure, one from the $[d\overline{\Psi}]$ and the other from $[d\Psi]$.) As a consequence, we derive the anomalous equation for the axial vector current, which in 2 dimensions is

$$\partial_\mu \overline{\Psi}\gamma^\mu\gamma_5\Psi - 2mi\overline{\Psi}\gamma_5\Psi = -ig^2/(2\pi)\ \epsilon \cdot F \tag{39}$$

and in 4 dimensions

$$\partial_\mu \overline{\Psi}\gamma^\mu\gamma_5\Psi - 2mi\overline{\Psi}\gamma_5\Psi = -ig^2/16\pi^2\ FF^d \tag{40}$$

The factor of i in both these expressions may be removed by analytically continuing back to Euclidian space.

For the baryon plus lepton number current anomaly in the Weinberg-Salam model, the analysis of the anomaly is a little more complicated by the left handed nature of the interaction with the gauge field. In the analysis presented in this lecture, we shall consider the Weinberg-Salam model in the limit that the Weinberg angle is zero $\Theta_W = 0$. We shall therefore ignore electromagnetic interactions in our analysis, that is we consider a non-abelian gauge theory of left handed quarks and leptons interacting with SU(2) vector fields. We could include the effects of the electromagnetic interactions if we chose to, but it complicates our analysis, and it shall be deleted. It causes no essential change in our conclusions.

If we make a U(1), B+L rotation of the fermion fields, the action changes by

$$\delta S = (\partial_\mu \alpha(x))\overline{\Psi}\gamma^\mu\Psi \tag{41}$$

The Higgs and vector meson couplings are clearly invariant under this transformation, and make no contribution to the change in the action. Also, the right handed fermion fields in the path integral measure do not couple to the gauge fields, and therefore when the change in the measure due to these right handed fields is computed, there are no contributions. It is now a straightforward repetition of the analysis done above to prove that the change in the measure is

$$d\mu \rightarrow d\mu\ exp\{iN_f g'^2/16\pi^2 \int d^4x\ \alpha(x) F^{\mu\nu}_\alpha F^{\alpha\ d}_{\mu\nu}(x)\} \tag{42}$$

The factor in the exponential is 1/2 what it would have been for the case of a pure vector theory. Here N_f is the number of fermion generations.

The anomaly in the baryon plus lepton number number current becomes

$$\partial_\mu J^\mu_{B+L} = N_f g'^2/16\pi^2 \; FF^d \tag{43}$$

(If we had included the contribution from the electromagnetic fields, we would have also included a $FF^d + F_{em}F^d_{em}$ term on the right hand side of the previous equation.)

The contribution FF^d may also be written as a total divergence,

$$\frac{g^2}{32\pi^2} FF^d = \partial_\mu K^\mu \tag{44}$$

The Chern-Symons number current is here

$$K^\mu = \frac{g^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}\{F^\alpha_{\nu\lambda}W^\alpha_\sigma - \frac{2}{3}g'\epsilon^{\alpha\beta\gamma}W^\alpha_\nu W^\beta_\lambda W^\gamma_\sigma\} \tag{45}$$

where W is the W boson field. The current

$$J_{B+L} - 2N_f K \tag{46}$$

is conserved, but gauge dependent. The total Chern-Symons charge

$$Q = \int dt K^0 = \frac{g^2}{32\pi^2}\int d^4x FF^d \tag{47}$$

is non-zero in Euclidian space for those fields with non-trivial topology, that is, instantons. Other fields give no contribution to Q. The integral of the divergence of the Chern-Symons current, which is proportional to FF^d is of course gauge invariant, and this measures the amount of baryon number production.

Notice that in this derivation, no anomaly can arise in the $B - L$ number current. This follows because the change in the measure factor associated with a chiral rotation for the fermions cancels that for the bosons.

2.2 A Physical Interpretation of the Anomaly

The chiral anomaly relates the current of fermions to external gauge fields. It requires that in the presence of classes of external fields, the fermionic axial charge, or in the case of the Weinberg-Salam model B+L, is not conserved. It is useful to have a physical intuitive picture of how this actually takes place.

Qualitatively, we can understand the anomaly by studying what happens to the Fermi sea in the presence of an external field. Consider the case of massless fermions. The Fermi sea is labeled by eigenstates of γ_5. In Fig. 3, a half filled Fermi sea of states with $\gamma_5 = \pm 1$ is shown. Due to the the chiral invariance of the equations of motion, for every state of positive chirality $\gamma_5 = 1$, there is a partner of negative chirality.

Now suppose we try to make a transition which produces a particle of positive chirality together with an anti-particle, that is a hole in the Fermi sea of negative chirality. This corresponds to pseudo-scalar meson production, and is forbidden by the chiral invariance of the equations of motion.

We may however try to do this by turning on an external field, and adiabatically shifting the energy levels. We might turn on an external vector potential which raises the energy of positive chirality states, and decreases those of negative chirality. Then it is possible that the occupation of the energy levels may change. For example if by applying an external field we may make a transition of one of the negative chirality states into a state of positive chirality, we may end up with the situation shown in Fig. 4.

We just argued however that such a situation, which corresponds to pseudo-scalar meson production is not allowed by the equations of motion. How is it that we succeeded in making the pseudo-scalar meson? The way this happens is because in theories which after quantization and regularization maintain the chiral symmetry there is a doubling of the spectra. This happens for example in the lattice, where the energy is related to the momentum by $E \sim \sin k$. There are two low energy states, corresponding to a state with $k \sim 0$ and a state with $k \sim \pi$. The second state is at the bottom of the fermi sea, and still has small energy. In this case, when an external field is applied, the states of small momentum do as described above. However, the states of momentum $k \sim \pi$ do the opposite, that is, there is a negative chirality particle and positive chirality hole created. Therefore, we produce a pseudo-scalar meson and its parity doubled partner, and therefore, therefore U(1) chiral invariance is maintained. The basic point is that here chiral invariance does require a doubling of the spectra, but the chiral doubled partners are at opposite ends of the Fermi sea.

In realistic theories, we do not wish to have an energy momentum spectral relation where large momentum particles have small energy. Therefore, realistic regularization schemes invariably break chiral U(1). They have the effect of making after regularization the contribution from states of large momentum ignorable. Therefore, after regularization, our naive arguments about level crossings in external fields do in fact lead to the conclusion that chiral symmetry is broken. This observation leads to the physical origin of the U(1) anomaly.

I shall now derive the U(1) anomaly from these physical considerations using the example of 1+1 dimensional massless QED. I will follow closely the beautiful analysis of Ambjorn, Greensite and Peterson.[16] The action for this theory is

$$S = \int d^2x \overline{\Psi}\{\gamma \cdot (-i\partial - eA)\}\Psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{48}$$

In the analysis that follows, we shall work in the gauge

$$A^0 = 0 \tag{49}$$

The gamma matrices for 1+1 dimensional fields theory have been described above.

In the analysis that follows, it will be useful to regularize the axial vector current by split pointing the current. We shall first check to see that this regularization method reproduces the results of Fujikawa's analysis, as it must. I let ϵ be a parameter $\epsilon \ll 1$ which in the end of our computations we take to zero. The split point chiral U(1) split charge is

$$Q_5(t) = \int dxdy\ \delta_\epsilon(x-y)\ \Psi^\dagger(x,t)\gamma_5\Psi(y,t)\ exp\{ie\int_x^y dx' A_1(x',t)\} \tag{50}$$

with

$$\delta_\epsilon = \frac{1}{2\pi\epsilon}e^{-x^2/2\epsilon} \tag{51}$$

This definition provides a gauge invariant method for splitting the points apart in the two fermion fields which define the current operator. This splitting makes the definition of the chiral charge finite and well defined for any finite value of ϵ.

We may now use the equations of motion on this expression for the split point current. We find

$$\frac{d}{dt}Q_5(t) = ie\int dxdy\ \delta_\epsilon(x-y)\ (x-y) \tag{52}$$

$$\{-\partial_x A_x\ \Psi^\dagger(x,t)\Psi(y,t) + \partial_t A_x\ \Psi^\dagger(x,t)\gamma_5\Psi(y,t)\} \tag{53}$$

Now for matrix elements of fermion fields in the vacuum, as finite temperature expectation values, or as matrix elements between the in and vacuum in external fields, the short distance singular behavior of these operators is determined by the vacuum expectation value. Now

$$\langle 0 \mid \Psi^\dagger(x,t)\Psi(y,t) \mid 0\rangle = \int d^2k/(2\pi)^2 \; tr \; \gamma^0(\frac{1}{\gamma \cdot p}) \; e^{ik_1(x_1-y_1)} \tag{54}$$

$$= 0. \tag{55}$$

On the other hand, for the other expectation value, we have

$$\langle 0 \mid \Psi^\dagger(x,t)\gamma_5\Psi(y,t) \mid 0\rangle = \int d^2k/(2\pi)^2 \; tr \; \gamma^1(\frac{1}{\gamma \cdot p}) \; e^{ik_1(x_1-y_1)} \tag{56}$$

$$= \frac{i}{\pi} \frac{1}{(x_1 - y_1)} \tag{57}$$

We find therefore in the limit that $\epsilon \to 0$ that the axial charge satisfies the anomaly equation

$$\frac{d}{dt}Q_5(t) = \frac{e}{2\pi}\int dx F^d(x,t) \tag{58}$$

which is the same result derived in Fujikawa's method.

Now we shall consider massless Dirac fermions in the presence of an external field. The purpose of this exercise will be to demonstrate that the number of particles which cross levels in the presence of this external field is precisely the value given by the chiral U(1) anomaly. The field we shall consider is

$$A_1(x,t) = \Theta(t)A \tag{59}$$

We expand the Dirac field in terms of creation and annihilation operators as

$$\Psi(x,t) = \int \frac{dk}{2\pi} e^{ikx}\{u_k(t)b_k + v_{-k}(t)d^\dagger_{-k}\} \tag{60}$$

The Dirac equation for the spinors u and v is

$$i\partial_t u_k = \gamma_5(k - eA(t))u_k \tag{61}$$

$$i\partial_t v_{-k} = \gamma_5(k - eA(t))v_{-k} \tag{62}$$

For $t \leq 0$, the spinors in the previous equation are given by

$$u_k(t) = e^{-i\omega_k t}u_k^{(0)} \equiv u_k^{(0)}(t) \tag{63}$$

$$v_k(t) = e^{i\omega_k t}v_{-k}^{(0)} \equiv v_{-k}^{(0)}(t) \tag{64}$$

Here $\omega_k = \mid k \mid$. Notice that u and v are helicity eigenstates

$$\gamma_5 u_k^{(0)} = sign(k)\, u_k^{(0)} \tag{65}$$

$$\gamma_5 v_k^{(0)} = sign(k)\, v_k^{(0)} \tag{66}$$

For time $t \geq 0$, the energy levels labeled by k become shifted as

$$E_k = \mid k \mid - e\, sign(k) A \tag{67}$$

As shown in Fig. 5a, the states of positive and negative helicity are shifted as shown in Fig. 5b.

Those positive helicity and positive energy states with energy $0 \leq E_k \leq eA$ are now occupied, and negative energy states with energy $0 \geq E_k \geq -eA$ are now unoccupied. Compared to the vacuum in the absence of an external field, there are now particles and anti-particles present. To count the number of such states, we imagine the system is in a box of length L. The spacing in momentum between these states is therefore $2\pi/L$. The chirality change is the number of particle plus the number of holes created. We find

$$\delta Q_5 = 2eA/(2\pi/L) \tag{68}$$

$$= \frac{e}{\pi} \int_0^L dx \int \partial_t A_1(t) \tag{69}$$

$$= \frac{e}{2\pi} \int d^2x\, F^d \tag{70}$$

This is precisely the expression for the anomaly as derived by the method of Fujikawa.

We can see this anomaly also directly arise from our point split definition of the chiral charge. In this case, we evaluate the charge in momentum space for time t after the turn on of the external vector potential. We obtain

$$\langle 0_{in} \mid Q_5 \mid 0_{in} \rangle = L \int \frac{dk}{2\pi} e^{-\frac{1}{2}\epsilon(k-eA)^2} \tag{71}$$

$$\langle 0_{in} \mid b_k^\dagger b_k u_k^\dagger(t) \gamma_5 u_k(t) + d_{-k} d_{-k}^\dagger v_{-k}^\dagger(t) \gamma_5 v_{-k}(t) \mid 0_{in} \rangle \tag{72}$$

$$= L \int \frac{dk}{2\pi} e^{-\frac{1}{2}\epsilon(k-eA)^2} v_{-k}^\dagger(t) \gamma_5 v_{-k}(t) \tag{73}$$

$$= L \int \frac{dk}{2\pi} e^{-\frac{1}{2}\epsilon(k-eA)^2} sign(-k) \tag{74}$$

$$\tag{75}$$

Now we can see the nature of the possible singularity of dealing with an unregulated theory. If we for example in the previous integral first took the limit $\epsilon \to 0$ before doing the integral, then the above integral would be of the form $+\infty$ from the upper half integration range and $-\infty$ from the bottom half, and would therefore be undefined. We see however, that this integral is

$$
\begin{aligned}
\langle 0_{in} \mid Q_5 \mid 0_{in} \rangle &= \frac{L}{2\pi} \int_{-eA}^{eA} dk \; e^{-\frac{1}{2}\epsilon(k-eA)^2} && (76) \\
&= \frac{L}{2\pi} 2eA && (77) \\
&= \delta Q_5 && (78)
\end{aligned}
$$

We could carry out precisely this same kind of analysis for the 4 dimensional gauge theory, although the conclusions and analysis are here essentially the same. There is a potential problem with interpreting the change in the axial vector current in terms of level crossing when there is radiation in the final state. As has been shown by Christ [17], the anomaly is still true. However, what happens is that instead of all the anomally being associated with a levels which have crossed, some of the anomally associated with radiation fields has to be defined to be part of the vacuum charge and is hidden in a normal ordering term in the definition of the chiral charge. This is a consequence of the fact that at late times, the fermion fields are propagating in the presence of the radiation field, and in some sense cannot be thought of as true asymptotic states. (If the vector potential is a constant at large times as was the case above, then the field tends to a gauge transformation of the vacuum, and the asymptotic states are in this sense well defined). We shall not further concern ourselves with this subtlety here.

3 Lecture 2: Topology and Energy Barriers

3.1 Introduction

In this section, we shall analyze some topological aspects of various theories. In particular, we shall consider systems with multiply periodic potentials, that is where there are degenerate minima. Conventional perturbation theory is done as an expansion around one of these minima. We shall here discuss how these multiple minima are classified, as well as the energy barrier which separates these minima. Following Manton, we construct a topological argument which proves the existence of this barrier. We construct classical unstable static solutions of the equations of motion (sphalerons) which allow an evaluation of the height of this barrier.

The relation between topology and stationary phase points may be seen in a simple example. Consider the torus shown in Fig. 6. We can consider height as a function of position on the surface of the torus. Because the torus is compact, there must be a minimum at the point P_0 and a maximum at the point P_3. This is not surprising since any compact object, for example a two sphere, must have such a maximum and a minimum.

What is surprising about the torus is that the topology of the torus also requires the existence of two stationary phase points, denoted by P_1 and P_2. To construct a topological argument, consider the class of loops, an example of which is shown by the dashed line in Fig. 6, which thread around the torus and are required to pass through the point P_0. On each such loop, there is a maximum of height. The minimum value of all such possible maxima gives the height of the point P_1. Therefore, there must exist a stationary phase point P_1 which is a maximum as one passes around the loop, and this is the maximum which is the minimum of all maxima, of a variable which parameterizes different loops.

In field theory models, the classification of different possible minima, and the barriers between them is useful since we may in general make transitions between these different minima. We shall later see that in the electro-weak theory for example, the transition between topologically distinct minima of the effective potential give changes in baryon number.

The pendulum model is a useful example of a system with many local minima. These minima are characterized by the angular variable of the pendulum as shown in Fig. 2. If we attempt to deform the coordinate of the pendulum from one of its minima to another, we must pass through a local maximum in energy, and this energy is the height of the barrier between the two local minima. There is a static unstable classical solution of the equations of motion along the path of deformation which corresponds the top of the barrier. We call this solution the sphaleron. We have

$$E_{barrier} = V(x_{sphal}) \tag{79}$$

This will be generally the case in field theory models. When there is a stationary phase point of the energy, then there should also exist static, unstable solutions of the classical equations of motion, sphalerons. If it can be argued that the sphaleron lies on a path of deformations of classical fields which connect two minima of the effective potential theory with different topological characteristics, then the sphaleron gives the height of the energy barrier between these minima.

3.2 The O(3) Sigma Model in 1+1 Dimensions

The 1+1 dimensional O(3) sigma model provides the simplest field theory model for topological charge changing processes.[18],[19]–[20] (There has been an analysis similar to the one of Mottola and Wipf for the U(1) Higgs model in 1+1 dimensions, with similar conclusions.[21]) The action for this theory is

$$S = \frac{1}{2g^2}\int d^2x(\partial_\mu \hat{n}\cdot\partial_\mu \hat{n}) \tag{80}$$

The field $\hat{n}$ is a unit 3 vector

$$\hat{n}^2 = 1 \tag{81}$$

This theory bears a remarkable resemblance to gauge theories in 3+1 dimensions. In particular, this theory is scale invariant and asymptotically free. At high temperatures, a mass gap of order g^2T develops for the scalars, in analogy to the situation in non-abelian gauge theories for the magnetic correlation length.[9],[22]

Finally, and most importantly, there is an anomaly in this theory for fermions. To see this, we may extend this model in a supersymmetric way, including Majorana fermions in the adjoint representation.[23]–[24] The action for this theory is

$$S = \frac{1}{2g^2}\int d^2x\ \{\frac{1}{2}(\partial_\mu n_a)^2 + \frac{1}{2}\overline{\Psi}_a(-i\gamma\cdot\partial)\Psi_a + \frac{1}{8}(\overline{\Psi}\Psi)^2\} \tag{82}$$

with the constraints that

$$n^a n^a \ = \ 1 \tag{83}$$

$$n_a\Psi_a \ = \ 0 \tag{84}$$

It is possible to show that in this theory that the axial vector current,

$$J_5^\mu = \epsilon^{ijk}\hat{n}_i\overline{\Psi}_j\gamma^\mu\gamma_5\Psi_k \tag{85}$$

is not conserved

$$\partial_\mu J_5^\mu = \frac{ig^2}{\pi}\epsilon_{\mu\nu}\hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) \tag{86}$$

The quantity

$$Q = \frac{1}{8\pi}\int d^2x\ \epsilon_{\mu\nu}\ \hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) \tag{87}$$

is the topological charge for this theory. The topological charge density may also be written as the divergence of a current

$$\partial_\mu K^\mu = \epsilon_{\mu\nu}\ \hat{n}\cdot(\partial_\mu\hat{n}\times\partial_\nu\hat{n}) \tag{88}$$

where we can choose

$$K^\mu = 2\epsilon^{\mu\nu}\ cos\Theta\ \partial_\nu\Phi \tag{89}$$

We have here written the unit vector $\hat{n}$ as

$$\hat{n} = (sin\Theta cos\Phi, sin\Theta sin\Phi, cos\Theta) \tag{90}$$

There are of course instantons in this model[18]. The instantons have topological charge n, and in terms of the redefined variables

$$z = t + ix \tag{91}$$

and

$$w = \frac{n_1 + in_2}{1 + n_3} \tag{92}$$

are the simple meromorphic functions

$$W_n = c\ \Pi_{j=1}^n\ \frac{z - a_j}{z - b_j} \tag{93}$$

Anti-instantons have $z \to \overline{z}$. The action (+) and the topological charge (-) are in terms of w given as

$$\frac{4}{g^2}\int d^2x\ \frac{1}{(1+\mid w\mid^2)^2}\ \{\frac{\partial w}{\partial z}\frac{\partial\overline{w}}{\partial\overline{z}} \pm \frac{\partial w}{\partial\overline{z}}\frac{\partial\overline{w}}{\partial z}\} \tag{94}$$

The above expressions for instantons are valid at zero temperature. At finite temperature, care must be taken to maintain periodicity in Euclidian time under $t \to t + \beta$. Single instanton solutions are nevertheless easy to construct by taking $a_j = a + j\beta$ and $b_j = b + j\beta$. The result of doing this is

$$W = c\ \frac{sinh\pi(z-a)/\beta}{sinh\pi(z-b)/\beta} \tag{95}$$

The existence of Euclidian space equations of motion for this theory should be expected. The gauge group is O(3), corresponding to the two sphere S_2. The two dimensional manifold may also be parameterized as S_2 by a stereographic projection of the variables in the plane into the angular variables of the sphere. The winding number of the instanton corresponds to the mappings of $S_2 \to S_2$.

Ordinarily, instanton solutions are interpreted as tunneling solutions between degenerate vacua. (Solutions of classical Euclidian equations of motion correspond to tunneling processes in the WKB approximation). We shall here explicitly construct a path of non-trivial topology which allows for a transition over a barrier between these vacua. If we view the parameter which labels that path μ as time dependent, then the path so constructed will make a transition by one unit of topological charge, and therefore corresponds to a process which if dynamically realized would change chiral fermion number.

We must be a little careful here about what we mean by transitions between different vacua. What we precisely mean here are changes in classical fields between a classical field oriented uniformly in one direction to the same field by intermediate fields which change winding number. A uniformly oriented classical fields has zero classical field energy. However, as a consequence of Coleman's theorem, in 1+1 dimensions, the ground state has a disordered ground state, and the expectation value of the classical field must vanish. We shall cure this later by introducing a term into our action which explicitly breaks the rotational symmetry, and allows therefore for an expectation value for the field.

Consider the following mapping of the configuration space to vector $\hat{n}$:

$$\hat{n} = (sin\mu sin\theta, sin^2\mu cos\theta + cos^2\mu, sin\mu cos\mu(cos\theta - 1)) \tag{96}$$

This mapping satisfies

- $\hat{n}^2 = 1$, and is continuous in its arguments
- for fixed μ, θ is the azimuthal angle of a circle S_1
- we have

$$\begin{aligned} \hat{n}(\mu, \theta = 0) &= \hat{n}(\mu, \theta = 2\pi) \qquad (97) \\ &= (0, 1, 0) \qquad (98) \end{aligned}$$

$$\hat{n}(\mu = 0, \theta) = \hat{n}(\mu = \pi, \theta) \tag{99}$$

$$= (0, 1, 0) \tag{100}$$

- each point on S_2 occurs for at least one (μ, θ), and if $\hat{n}$ is not the point $(0, 1, 0)$, then $\mu(\hat{n})$ is unique.
- as μ ranges from 0 to π and θ from 0 to 2π, this mapping has topological winding number 1.

The first four items above are essentially obvious, and easy to show. To show the fifth point, it is straightforward to show that the mapping above comes from the points of intersection between a plane and a 2-sphere as shown in Fig. 7. This figure clearly shows that the mapping corresponds to looping a 1-sphere around the two sphere, and we expect it has non-trival winding number. If we let μ and θ parameterize our configuration space and let $\mu = \mu(t)$, the topological charge density is

$$\epsilon^{\mu\nu} \frac{1}{8\pi}\, \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \frac{1}{4\pi} sin\mu (1 - cos\theta) \tag{101}$$

Integrating this over all theta, and between $0 \leq \mu \leq \mu'$ gives

$$\delta Q = \frac{1}{2}(1 - cos\mu') \tag{102}$$

For $\mu' = \pi/2$, the topological charge is 1/2. In the initial and final configurations, we have a uniformly oriented field configuration. We would naively expect that the top of the barrier corresponds to 1/2 unit topological charge.

For the $1 + 1$ dimensional O(3) sigma model, it is difficult to construct an explicit sphaleron solution which corresponds to the top of the barrier. Moreover, as a consequence of Coleman's theorem, we can never spontaneously break a continuous symmetry in $1 + 1$ dimensions, so saying that the vacuum corresponds to a field pointing in some specified direction is without content. To be able to analyze these aspects of the problem, it is convenient to modify the O(3) model. We do this by explicitly breaking the symmetry, and modifying the action to

$$S = \frac{1}{g^2} \int d^2x \; \{\frac{1}{2}(\partial n)^2 + \omega^2(1 - \hat{n}_2)\} \tag{103}$$

Adding this last term top the action breaks the scale invariance of the action, and therefore allows for the existence of static solutions which are stable under scale transformations (which may be unstable under other modes), and will therefore allow the sphaleron to exist.

For temperatures

$$T \ll \omega/g^2 \tag{104}$$

the mass gap generated for the scalar boson propagator is sufficiently large so that weak coupling methods are reliable. To see this, suppose we are at high enough temperature so that the theory becomes effectively one dimensional. That is, in the action it becomes a good approximation to take $\int_0^\beta dt \sim \beta$, and ignore time derivatives. Then, the action becomes

$$S = \frac{\omega\beta}{g^2}\int dx\ \{\frac{1}{2}(\partial n)^2 + (1-n_2)\} \tag{105}$$

Here we have rescaled the spatial coordinate x by ω to make it dimensionless. It is now clear that the coupling constant of this one dimensional theory is

$$g'^2 = g^2/(\beta\omega) \tag{106}$$

and weak coupling techniques are valid when $T \ll \omega/g^2$. At temperatures larger than this, the relevant scale in the problem becomes the magnetic mass, and then

$$g'^2 = g^2/(\beta g^2 T) \sim 1 \tag{107}$$

so that perturbation theory is never valid.

This is parallel to the situation in the electro-weak theory. At temperatures

$$M_W \ll T \ll M_W/g^2 \tag{108}$$

weak coupling techniques can be shown to be reliable by a scaling argument identical to the one given above for the O(3) sigma model in 1+1 dimensions. At higher temperatures, infrared singularities are cutoff by a magnetic mass of order g^2T, which however invalidate the use of weak coupling methods.

We can now easily construct the sphaleron of the O(3) sigma model. Consider the mapping above where we allow $\theta = \theta(x)$. The analysis of the topological charge goes through as before, and such a field has topological charge of $(1 - cos\mu)/2$. We can now minimize the energy with

$$\hat{n} = (sin\theta(x), cos\theta(x), 0) \tag{109}$$

The action for this configuration is

$$S = sin^2\mu/g^2 \int dx\ \{\frac{1}{2}(d\theta/dx)^2 + \omega^2(1 - cos\theta)\} \tag{110}$$

This action, which is the same as the energy for a static configuration, has a maximum at $\mu = \pi/2$, corresponding to being at the top of a barrier for changing topological charge. Extremizing the action yields the sphaleron

$$\theta_{sphal}(x) = 2sin^{-1}(sech\omega x) \tag{111}$$

and the energy of the height of the barrier

$$E_{sphal} = 2\omega/g^2 \tag{112}$$

Although we have found an explicit sphaleron solution in this theory, we are not guaranteed that there might not exist other solutions with a lower barrier height. Therefore, at best we can only be convinced that by finding such a solution we get an upper bound on barrier height, which we shall later see corresponds to a lower bound on transition rates. Finally, unless small fluctuations are carried out around the the sphaleron, the sphaleron itself is of zero measure in its contribution to the path integral. We cannot a priori rule out the possibility that summing over such fluctuation gives either a zero or numerically very small result.

3.3 Topology and Sphalerons in Electro-Weak Theory

In electro-weak theory, we can classify topologically inequivalent vacuum field configurations. Recall that if we have any field configuration, then a gauge transform of that field configuration has identical energy. There are gauge transformations which can be continuously connected to the identity transformation, and for fields which are such transforms, we can fix a gauge by a continuous transform of the fields.

There may be transformations which cannot be continuously shrunk to the identity however. For SU(2) gauge theory in 3+1 Euclidian dimensions, we can see immediately that this is the case. If we consider gauge transformations which for large spatial $|\vec{x}|$ behave as

$$lim_{|\vec{x}| \to \infty}\ g(x) \to 1 \tag{113}$$

then the set of points, E_3 of $\vec{x}$, may be mapped into S_3 by a stereographic projection. On the other hand the group manifold of SU(2) is also S_3. Therefore, there should be a mapping of $S_3 \to S_3$ with a winding number n. Gauge transformations with non-zero winding number may not be continuously distorted into the identity element.

The energy of SU(2) gauge theory as a function of the fields therefore has a structure analogous to that of the pendulum as a function of the angular coordinate of the pendulum, Fig. 2. As a function of the fields, there must be a set of local degenerate minima corresponding to topologically inequivalent lowest energy field configurations. The fields may be distorted between one minima and another, and in general there is a barrier between these minima. The labels in each minima are not of physical significance, since the fields there are in fact pure gauge.

There is a topological charge associated with the gauge theory. It is

$$Q = \frac{g^2}{32\pi^2} \int d^4x \; F F^d \tag{114}$$

and a Chern-Symons current

$$\partial_\mu K^\mu = \frac{g^2}{32\pi^2} F F^d \tag{115}$$

The value of the Chern-Symons charge is gauge dependent but clearly the integral of its four divergence is gauge independent, and it is this change in Chern-Symons charge which measure the amount of $B + L$ violation,

$$\partial J_{B+L} = 2N_f \partial K \tag{116}$$

(The relation between the amount of $B+L$ violation and the Chern-Symons charge was discussed in the first lecture)

There are no instanton solutions in the broken symmetry phase of electro-weak theory. This is because of the non-trivial vacuum expectation value for the Higg's field. There are of course field configurations with non-trivial winding number. Also, as a consequence of the vacuum structure and the topological classification discussed above, we expect that there may be transitions between these different vacuum field configurations.

We can provide an estimate for the action of field configuration with non-zero topological charge. Recall that in the electro-weak theory, ignoring electromagnetism as we shall do throughout these lectures,

$$S = \int d^4x \, \{\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + (D_\mu\Phi)^\dagger(D^\mu\Phi) + V(\Phi)\} \tag{117}$$

Here the Yang-Mills part of the action is for SU(2) vector fields. The Higgs field is a complex SU(2) doublet, and the potential $V(\Phi)$,

$$V(\Phi) = \lambda(\Phi^\dagger\Phi - \frac{v^2}{2})^2 \tag{118}$$

has a minimum at a non-zero value of Φ which we take by convention to be

$$\Phi = \frac{v}{\sqrt{2}}u_{-\frac{1}{2}} \tag{119}$$

where

$$\sigma_3 \, u_{-\frac{1}{2}} = -1 \tag{120}$$

is an elementary isospin $-1/2$ spinor. We have here ignored the fermion contributions to the action.

The contribution to the scalar fields are always positive. Therefore a lower bound for the action is provided by the SU(2) vector field strength. On the other hand, we have

$$(F \pm F^d)^2 = 2(F^2 \pm FF^d) \tag{121}$$

so that the lower bound is

$$S \geq \frac{8\pi^2}{g^2}Q \tag{122}$$

Since $\alpha' = g^2/4\pi \sim 1/40$, we expect that the contributions of such configurations to the Euclidian path integral should be small.

We can now study the stationary phase point which separates topologically distinct minima of the electro-weak theory. We begin by noting that the Higgs field is of the form

$$\Phi = \begin{pmatrix} Re\Phi_1 \\ Im\Phi_1 \\ Re\Phi_2 \\ Im\Phi 2 \end{pmatrix} \tag{123}$$

Now as in the case of the O(3) sigma model, we make a topologically non-trivial map from the spatial manifold onto the field space as

$$\Phi(\mu,\theta,\phi)=\begin{pmatrix} sin\mu sin\theta cos\phi \\ sin\mu sin\theta sin\phi \\ sin^2\mu cos\theta+cos^2\mu \\ sin\mu cos\mu(cos\theta-1) \end{pmatrix} \tag{124}$$

As was the case for the sigma model, the set of points

$$x(\mu,\theta,\phi)=(sin\mu sin\theta cos\phi, sin\mu sin\theta sin\phi, sin^2\mu cos\theta+cos^2\mu, sin\mu cos\mu(cos\theta-1)) \tag{125}$$

corresponds to the intersection of a plane with a three sphere. This mapping is clearly topologically non-trivial, and takes the vacuum expectation value of the Higgs field at $\mu=0$ into the vacuum expectation value at $\mu=\pi$.

We can write

$$\Phi(\mu,\theta,\phi)=Vu_{-\frac{1}{2}} \tag{126}$$

It is straightforward to show that

$$V=\begin{pmatrix} e^{i\mu}(cos\mu-isin\mu cos\theta) & sin\mu sin\theta e^{i\phi} \\ -sin\mu sin\theta e^{-i\phi} & e^{-i\mu}(cos\mu+isin\mu cos\theta) \end{pmatrix} \tag{127}$$

The matrix V can be thought of as a gauge transformation. We can write the vector field resulting from this gauge transformation as

$$\tau\cdot A_\mu=(\frac{i}{g}\partial V)V^{-1} \tag{128}$$

To see that a field configuration with this topology has a non-trivial winding number, we must compute the topological charge. We let the parameter $\mu(t)$, where we assume that $\mu(-\infty)=0$ and $\mu(\infty)=\pi$. We also need r dependence of the field configuration so that it will have finite energy. We here will let the field configuration be

$$\Phi(\mu,r,\theta,\phi)=[1-h(r)]\begin{pmatrix} 0 \\ e^{-i\mu}cos\mu \end{pmatrix}+h(r)Vu_{-\frac{1}{2}} \tag{129}$$

$$A=f(r)\frac{i}{g}\partial VV^{-1} \tag{130}$$

Requiring the solution to have finite energy implies

$$lim_{r\to 0}\, h(r) = 0, \quad lim_{r\to\infty}\, h(r) = 1 \tag{131}$$

$$lim_{r\to 0}\, f(r)/r = 0, \quad lim_{r\to\infty}\, f(r) = 1 \tag{132}$$

Notice that for $\mu = 0$ and for $\mu = \pi$, these fields are those of the vacuum.

We can now compute the topological charge density. For $0 \le \mu \le \pi$, after a good deal of algebra, one can prove that $0 \le Q \le 1$. At $\mu = \pi/2$, the topological charge is $1/2$.

It may also be shown that the ansatz above for the fields provides a solution of the equations of motion. This is non-trivial because of the assumed spherical symmetry of the fields f and h. The energy function has its maximal value at $\mu = \pi/2$. The solution at this value of μ is therefore the sphaleron of the electroweak theory. It is the classical solution half way between two different topological sectors of the theory. (We have not proven here that this solution truly corresponds to the stationary phase point. However, it is possible to do this.)

At $\mu = \pi/2$, the energy functional is

$$E \;=\; \int d^3x \frac{4}{g^2 r^2}\left(\left(\frac{df}{dr}\right)^2 + \frac{2}{r^2}(f(1-f))^2\right) \tag{133}$$

$$+\frac{v^2}{2}\left(\left(\frac{dh}{dr}\right)^2 + \frac{2}{r^2}(h(1-f))^2\right) + \frac{\lambda v^4}{4}(h^2-1)^2 \tag{134}$$

Minimizing this equation with respect to f and g gives the sphaleron solution.

The form we have written the sphaleron in is perhaps not the simplest for further analysis. The expression can be simplified a bit by making a gauge rotation and a custodial SU(2) right transformation. To understand the custodial SU(2) right transformations, we first note that ordinary SU(2) left transformations rotate the Higgs doublet as

$$\Phi_i \to \Phi_i + i\epsilon\tau_{ij}\Phi_j \tag{135}$$

The custodial $SU(2)_R$ transformation, which is also a symmetry of electroweak theory in the absence of electromagnetic interactions, is an SU(2) symmetry where the Higg's doublet is relabeled as

$$\begin{pmatrix} \Phi_1 \\ \overline{\Phi}_2 \end{pmatrix} \tag{136}$$

We can simplify our expression with the transformation

$$U_L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{137}$$

and

$$U_R = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{138}$$

The vector field is unchanged by the U_R transformation.

After such a transformation, the sphaleron is of the form

$$\vec{A} = 2v\frac{f(\xi)}{\xi}\hat{r} \times \vec{\tau} \tag{139}$$

and

$$\Phi = \sqrt{2}vh(\xi)\hat{r} \cdot \vec{\tau}u_{-\frac{1}{2}} \tag{140}$$

where

$$\xi = gvr \tag{141}$$

This form for a static solution of electro-weak theory was originally proposed by Dashen, Hasslacher and Neveu[25], and was discussed by Soni[26], and Boguta[27].

This ansatz for the sphaleron has the nice property that it is invariant under a combined rotation plus gauge transformation plus custodial $SU(2)_R$ transformation. It is not however invariant under rotations plus gauge transformations alone. To see this first notice that under a rotation plus a gauge transformation, the vector field can remain invariant. Under this same transformation, the scalar field rotates. This rotation may be undone by a $SU(2)_R$ rotation.

The sphaleron is not a stable solution of the electro-weak theory. To investigate stability, we can do small fluctuations in the presence of the sphaleron. Since the sphaleron is invariant under $R + SU(2)_R + SU(2)_L$, we find that the small fluctuations may be expanded in a partial wave expansion. We will not show, but it is indeed possible to show, that there is a single unstable mode which is a singlet under $R + SU(2)_R + SU(2)_L$.[20] All of the non-singlet modes and radial excitations of the singlet mode have positive semi-definite energy.

There are a set of six modes which are zero energy. Three of these correspond to translations of the sphaleron, and three to a $R + SU(2)_L$ transformation. Neither of these two transformations may change the energy of the sphaleron, so therefore the small fluctuations must give zero energy modes. In Appendix A, we discuss the construction of these zero modes.

The sphaleron may be easily constructed numerically. There are no known analytic solutions. The energy of these solutions may also be evaluated. For $0 \leq \lambda/g^2 \leq \infty$, the sphaleron energy is

$$E_{sp} = \frac{4\pi v}{g} A \tag{142}$$

where $1.52 \leq A \leq 2.70$. For $\lambda/g^2 = 1$, we have $A = 2.07$.

We can also directly evaluate the topological charge of the sphaleron. We already evaluated this in general, but it is useful to directly compute it here. In order that the only contribution to the Chern-Symons charge come from K^o, and not the divergence of $\vec{K}$, we work in a gauge where $\vec{K} \to 0$ at spatial infinity. We can do this by a non-singular gauge transformation,

$$U(\vec{r}) = exp[i\Theta(\vec{r})\vec{\tau} \cdot \hat{r}] \tag{143}$$

where $0 \leq \Theta \leq \pi$ as $0 \leq r \leq \infty$. We assume that at large r, $\Theta \to \pi$ sufficiently rapidly that the Chern-Symons current vanishes rapidly enough that there is no surface contribution to the topological charge. Under this gauge transformation, we have

$$A_i^a = \frac{[1-2f]cos\Theta - 1}{gr^2}\epsilon_{iab}r^b \tag{144}$$

$$+\frac{[1-2f]sin\Theta}{gr^3}(\delta_{ia}r^2 - r_i r_a) \tag{145}$$

$$+\frac{1}{g}\frac{d\Theta}{dr}\frac{r_i r_a}{r^2} \tag{146}$$

This expression for the vector potential may now be inserted into the expression for the Chern-Symons charge, with the result

$$Q_{CS} = \frac{1}{2} \tag{147}$$

We have seen in this lecture therefore that in electro-weak theory, there is a path which connects vacua of different topological charge, and hence baryon number. There is a path of deformation of the fields which connects these different minima. There is a barrier along the path of deformation, and an unstable static classical solution of the equations of motion which allows the computation of this height has been presented. This solution of the static equations of the motion has 1/2 unit of Chern-Symons charge, and is the sphaleron.

4 Lecture 3: The Sphaleron and Baryon Number Change at Finite Temperature

In the previous lecture, I showed that there exists a finite energy barrier between topologically distinct vacua, and the transitions between these vacua in the electro-weak theory induce a change in baryon number. Naively, we might expect that the rate of transition would be of order[12]

$$\Gamma/V \sim T^4 \kappa \; e^{-E_{sp}/T} \tag{148}$$

The factor of T^4 is to give the rate per unit volume the correct dimensions. The exponential factor is the requirement that states of the system have sufficient energy to transit over the top of the energy barrier. Notice that at low temperatures, this process of charge change decouples, and at high temperatures it decouples.

Of course to compute the rate, one must know the prefactor κ, and since this factor may have a non-trivial temperature dependence, the rate itself may be greatly modified. The rate itself might even be zero if by some circumstance $\kappa = 0$.

We can see that the factor κ may have a strong temperature dependence. At finite temperature, the Z-boson mass is temperature dependent,

$$M_Z = M_Z(0)\sqrt{1-(T/T_c)^2} \tag{149}$$

where T_c is the critical temperature of the electro-weak theory at which the SU(2) symmetry is restored.[28]–[30]

At high temperatures, by the scaling arguments presented in the last lecture, the only size scale that the sphaleron may depend upon is the Compton wavelength of the Z-boson. This assumes we are at temperatures $M_Z \ll T$, so that the Euclidian time integral in the action may be integrated out. We may or may not choose to work in the perturbative regime where $T \ll M_Z/g^2$. The only scale available to parameterize the sphaleron size is $1/M_Z$, and this diverges at the critical temperature. Therefore, if we were to make a dilute sphaleron gas approximation, where the rate is estimated by the decay of a single sphaleron, we might expect that the rate prefactor κ would have to vanish as $T \to T_c$. The sphaleron size has become much larger than a typical thermal wavelength.

For $T \geq T_c$, the situation is even more mysterious. Here there is no sphaleron solution at all. At such temperatures, of course the three dimensional coupling constant which parameterizes interactions for the high temperature theory

$$\alpha_3 = \frac{\alpha T}{M_Z(T)} \tag{150}$$

has diverged at $T = T_c$. Of course, for high temperatures, the coupling constant does not really diverge. Presumably the theory is cut-off at a size scale typical of a magnetic screening length, $d \sim 1/g^2T$.[22],[9] Due to the smallness of this magnetic screening length, perturbation theory has however broken down, and our semi-classical picture in terms of transitions across tops of energy barriers are no longer sensible.

In this lecture, I shall explore the formalism for computing finite temperature sphaleron induced transitions. In the first section, I review the analysis of Linde and Affleck,[31]–[32] for computing transition rates at finite temperature. In the second section, I discuss the application of this formalism for the computation of the transition rate in the O(3) sigma model. I analyze the transition process in the perturbative and non-perturbative regime. In the next section, I present the results of a similar analysis applied to the electro-weak theory, and estimate the rate for baryon number changing processes at high temperature. I show the rate per particle is 9-10 orders of magnitude faster than the expansion rate of the universe at a temperature just below that of the electroweak phase transitions, and argue the rate is larger at higher temperatures, even when the SU(2) symmetry is restored. In the last section, I address some confusion concerning an apparent discrepancy between the rate predicted by sphalerons and that for instantons.

4.1 Transitions at Finite Temperature

There are two regions where the formula for transitions from a metastable to a stable state, or between degenerate states, at finite temperature are simple. The first simple case is when the temperature is small compared to the mass scale of any particles in a theory. In this regime, the transition proceeds by quantum mechanical tunneling.

To understand the tunneling computation, consider the potential drawn in Fig. 8. At temperatures small compared to the frequency of oscillation at the bottom of the potential in the unstable minimum, we expect that there is only a very small probability for a state to make a real transition over the top of the barrier which separates the metastable state from the stable state. Therefore the process proceeds by tunneling under the barrier. The rate for each state to make a tunneling transition is $\Gamma(E) = 2\ ImE$. Taking the Boltzmann average, we find

$$\Gamma = 2\ ImF \tag{151}$$

where F is the free energy of the system.

It is straightforward to compute the free energy in a dilute gas of instantons. In a dilute gas, the free energy is the sum of all of the separated instantons, and exponentiates. In this way, we find

$$F/V \sim exp(-S_{inst}) \tag{152}$$

In the electro-weak theory, of course instantons are not true solutions of the equation of motion, and we cannot in general include their effects so trivially. Nevertheless, as discussed in previous lectures, for processes which involve topological charge change, it must be true that $S \geq S_{inst}$. For such processes at temperatures small compared to particle masses, we therefore have that the rate, up to a prefactor, is less than $exp(-S_{inst})$ The prefactor has been computed for the symmetric theory where instantons are true solutions of the equations of motion, and the prefactor cannot compensate the suppression due to instantons.[4]

The other simple limit is where the temperature is of order or large compared to the height of the barrier. When the temperature is large compared to the barrier height, we expect unsuppressed transitions. In this temperature range, in the electroweak theory, we will find however that it is not possible to compute the decay rate by weak coupling methods. However, if the temperature is $T \leq V_o$, this process may be computed by weak coupling methods. Here we simply compute the rate of transitions of states over the top of the barrier. In our later computations, we shall assume that $T >> M_Z$, since this considerably simplifies the analysis. In general however it may be possible that the rate of processes going over the top of the barrier may dominate over tunneling processes even for somewhat smaller temperatures.[34] It is obvious however that at some sufficiently small temperature, it must be true that the tunneling process becomes dominate, simply because the tunneling process does not turn off at zero temperature.

We can estimate the rate for going over the top of the barrier in Fig. 8 when $T \gg \omega_0$. In this limit, we expect that classical thermodynamics should give the correct result. The rate of making transitions over the top of the barrier is

$$\Gamma = \langle \delta(x - x_{barrier})\Theta(v)v \rangle \tag{153}$$

where v is the velocity operator. This expectation value is simply the flux of probability over the top of the barrier. We can compute this as

$$\Gamma = \frac{\int dpdx\ exp\left(-\beta[\frac{1}{2}p^2 + V(x)]\right)\ \delta(x - x_{barrier})v\Theta(v)}{\int dpdx\ exp\left(-\beta[\frac{1}{2}p^2 + V(x)]\right)} \tag{154}$$

$$\sim \frac{\omega_0}{2\pi}e^{-\beta V_o} \tag{155}$$

We have denoted the height of the barrier as V_o, and the frequency of small fluctuations around the metastable minimum as ω_0. To do the integral, we expanded the denominator above in Gaussian approximation.

We can now relate this expression for the rate to ImF. To do this we take the expression for the free energy and expand it around the top of the barrier, and around the minima in the metastable configuration.

$$ImF = TIm(lnZ) \sim T\frac{ImZ}{Z} \tag{156}$$

$$\sim T\frac{Im \int dpdx\ exp\left(-\beta[\frac{1}{2}p^2 + V_o - \frac{1}{2}\omega_- x^2]\right)}{\int dpdx\ exp\left(-\beta[\frac{1}{2}p^2 + \frac{1}{2}\omega_0 x^2]\right)} \tag{157}$$

$$\sim \frac{\omega_0}{2\omega_-\beta}e^{-\beta V_o} \tag{158}$$

In this equation, the factor of ω_- is the rate of decay in small fluctuations at the top of the barrier. The factor of $\frac{1}{2}$ comes from properly defining the analytic continuation of the integral over x so as to be convergent.

We now have a relation for the rate as

$$\Gamma \sim \frac{\omega_- \beta}{\pi} ImF \tag{159}$$

We shall use this relationship throughout our analysis. It should be carefully noted that this formula has been derived in Gaussian approximation. I know of no valid derivation beyond Gaussian approximation. (The derivation we have given here can be put on a more firm mathematical footing, as it was in its original derivation.) [33] In field theory, the Gaussian approximation is equivalent to weak coupling, and this will limit the quantitative range of validity of our results. For the electroweak theory, we shall be able to make quantitative conclusions in the narrow range of temperatures below T_c

$$M_Z(T) \ll T \ll M_Z(T)/\alpha' \tag{160}$$

It is true that for a temperature dependent mass, there is always a solution for this equations in a range of temperatures below T_c

The derivation for high temperatures must be extended now to systems with infinite numbers of degrees of freedom for it to be useful for field theory. In the Gaussian approximation for both Γ and ImF there is the same correction for each new degree of freedom

$$\frac{\int dp'dx' \ exp\left(-\beta[\frac{1}{2}p'^2 + \frac{1}{2}\omega'^2 x'^2]\right)}{\int dp'dx' exp\left(-\beta[\frac{1}{2}p'^2 + \frac{1}{2}\omega_0'^2 x'^2]\right)} = \frac{\omega_0'}{\omega'} \tag{161}$$

Therefore the relation between the rate and free energy is not affected.

For systems with infinite numbers of degrees of freedom, we can write down an explicit formula for the decay rate,

$$\Gamma \ \sim \ \frac{\omega_-}{\pi}\frac{Im \ Z_{barrier}}{Z_0} \sim \frac{\omega_-}{2\pi} Im \left(\frac{det_\beta \omega_0^2}{det_\beta \omega^2}\right)^{\frac{1}{2}} e^{-\beta V_0} \tag{162}$$

$$\sim \ \frac{\omega_-}{2\pi} Im \left(\prod \frac{sinh(\beta\omega_0^i/2)}{sinh(\beta\omega^i/2)}\right) e^{-\beta V_0} \to \frac{\omega_-}{2\pi} Im \left(\prod \frac{\omega_0^i}{\omega^i}\right) e^{-\beta V_0} \tag{163}$$

In the last line we have taken the limit $T \gg \omega$.

4.2 Transition Rate in the O(3) Sigma Model

We will now evaluate the transition rate for topological charge in the O(3) sigma model. We shall always work in the region where $T \gg \omega$, where ω was the symmetry breaking term in the action, in order that high temperature methods may be used. We shall consider two separate regions. The first is $T \ll \omega/g^2$, where weak coupling techniques may be used, and the second is $T \geq \omega/g^2$, where strong coupling analysis is required.

In the weak coupling region, we must compute the decay rate, which is given as the last formula of the previous section as

$$\Gamma \sim \frac{\omega_-}{2\pi} Im \left(\frac{det_\beta \omega_0^2}{det_\beta \omega^2} \right)^{\frac{1}{2}} e^{-\beta E_{sp}} \tag{164}$$

To compute the determinant, we need the modes of small fluctuations. We can get these modes by letting

$$\hat{n} = \frac{1}{\sqrt{1+u^2}} (sin(\xi_{sph} + v), cos(\xi_{sph} + v), u) \tag{165}$$

Using this in the O(3) sigma model action to quadratic order in small fluctuations gives the equations of motion

$$H_1 u = \left\{ -\frac{d^2}{dx^2} + \omega^2(1 - 6sech^2 \omega x) \right\} u = \lambda_1^2 u \tag{166}$$

$$H_2 v = \left\{ -\frac{d^2}{dx^2} + \omega^2(1 - 2sech^2 \omega x) \right\} v = \lambda_2^2 v \tag{167}$$

The operator H_1 corresponds to fluctuations which are in the direction of the loop variable which changes topological charge, and is orthogonal to the direction of the orbit of the sphaleron. We expect this operator to have a negative eigenvalue. There is also a zero mode associated with a rotation of the sphaleron. Explicit analysis confirms that these are in fact the only zero and negative mode for u. The operator H_2 has no negative mode, and a zero mode corresponding to translations of the sphaleron.

We could at this point compute explicitly the determinant of small fluctuations. This has been done by Mottola and Wipf.[20]. We can however extract all of the essential features of the result without ever doing the explicit sum over eigenvalues. These features follow from the result of Eq. 162-163.

To analyze the rate, we first must clean up one factor with which we have been a little sloppy. That is the factor of zero modes in Eq. 162. This has been discussed much in the literature, and we shall not repeat the formal analysis here. We only must notice that the determinants relevant here come from small fluctuation in the 1 dimensional theory, not in the 2 dimensional theory. Therefore, upon scaling all variables in the 1 dimensional theory so that they are dimensionless, scaling by ω, we have

$$\Gamma = \frac{\omega_-}{2\pi} g_1^{-N_0} \overline{NV} \left(\prod \frac{\omega_0}{\omega} \right) e^{-\beta V_o} \tag{168}$$

The product over frequencies has the zero modes deleted. Here the factor N_0 is the number of zero modes, which is two in this case, and g_1 is the coupling constant of the reduced dimensional 1 dimension theory

$$g_1^2 = Tg^2/\omega \tag{169}$$

The factor of $\overline{NV}$ is a dimensionless normalization integral for the zero modes, $\overline{N}$, and the volume of the symmetry group associated with the zero modes $\overline{V}$. Up to a constant, this group volume is the ordinary volume of space, V, times ω to make it a dimensionless volume. The frequency ω_- is a constant times ω by dimensional reasoning. The remaining product over frequency is dimensionless, and has no dependence on g^2, so it is a constant. We find therefore that the rate per unit volume is

$$\Gamma/V \ \sim \ \omega^2 g_1^{-2} e^{-\beta E_{sph}} \sim (g^2 T)^2 \alpha_1^{-3} e^{-\beta E_{sph}} \tag{170}$$

The corrections to this formula are constants with no dependence on T, ω or g.

The formula for the rate is written in a very suggestive way. The factor in front g^2T is the mass gap of the theory at high temperatures. What happens when the temperature becomes so large that $g_1 \sim 1$? Notice that $\beta E_{sph} \sim 1/\alpha_3^2$ so that this formula tends to M_{gap}^2. We will now show that this smoothly maps onto the result we expect for $T \gg \omega/g^2$

To understand the transition rate at high temperature, recall that the dynamics is described by a 1 dimensional field theory,

$$S = \frac{\omega\beta}{g^2} \int dx \ \{\frac{1}{2}(\partial n)^2 + (1 - n_2)\} \tag{171}$$

The 1 dimensional coupling constant is becoming large, so the system is becoming disordered. It is no longer a good approximation to imagine expanding around a single sphaleron, as the semi-classical limit is no longer relevant. A thermal coherence length $1/g^2T$ is much larger than a sphaleron size $1/\omega$

It is clear that as $T \to \infty$, the disordering of the system implies that topological charge changing processes are very probable. If we were to introduce a field into this system and give it a coherent twist, we expect it to dissapate in a time $1/g^2T$ Since transitions happen on a distance scale of order $1/g^2T$, we conclude that at asymptotic temperature, the rate of topological charge changing processes is

$$\Gamma/V \sim (g^2T)^2 \tag{172}$$

4.3 Baryon Number Change in the Electroweak Model

The analysis of baryon number violation in electroweak theory follows the lines which we have just analyzed for the O(3) sigma model. We will first work in the region where weak coupling methods are applicable

$$M_Z(T) \ll T \ll M_Z(T)/\alpha' \tag{173}$$

Actually, within this region there are two subregions. The first is

$$M_Z(T) \ll T \ll M_Z(T)/\sqrt{\alpha'} \tag{174}$$

In this region, the Debye screening length, the distance over which electric interactions are screened in a media,

$$R_{debye} = 1/\sqrt{\alpha'}T \tag{175}$$

is much larger than the size of the sphaleron. In this range of temperature, we may ignore Debye screening. It is this region for which I shall present results in this lecture.

For

$$M_Z(T)/\sqrt{\alpha'} \ll T \ll M_Z(T)/\alpha' \tag{176}$$

we are still in a regime where weak coupling techniques may be applied. There is a technical complication in the analysis of the decay of the sphaleron. Due to the Debye screening, it is difficult to establish an electric field in a media. Therefore, the sphaleron slows its decay to avoid setting up a singular field strength. This reduces the electric field energy since $E \sim 1/t$ where t is a characteristic decay time. The electric field energy is reduced as $\int dt\ E^2 \sim 1/t$. On the other hand the topological charge is unchanged by this slowdown since $\int dt\ EB \sim 1$.

The single sphaleron decay rate is affected by Landau damping and Debye screening in this temperature range.[13]. It is less clear how this affects the overall rate for topological charge changing processes.[35]. Perhaps the factors responsible for the density of sphaleron states cancel the suppresion due to the slowdown in the rate of decay of a single sphaleron. It is difficult to know because derivations of the transition rate rely heavily on Gaussian weak coupling methods, and when Debye screening becomes important these Gaussian approximations break down. In any case, in this lecture we address the computation of the rate when Debye screening is unimportant.

At high temperatures, the dynamics of electro-weak theory should be that of a three dimensional theory. Upon rescaling coordinates and fields to make them dimensionless,

$$(r,t) \to gv(r,t) \quad (A,\Phi) \to v(A,\Phi) \tag{177}$$

the electro-weak action becomes at high temperature

$$S_3 = \frac{1}{\alpha_3} \int d^3x \ L(A,\Phi,\lambda/g^2) \tag{178}$$

Here all explicit dependence on v has disappeared. The three dimensional coupling is

$$\alpha_3 = \alpha' \frac{T}{2M_Z} \tag{179}$$

In this derivation of the three dimensional effective theory, we have not been careful about defining the action S_3. The problem is that when quantum corrections are computed for the three dimensional theory, the four dimensional structure of finite temperature corrections to the three dimensional action are important. These come from ultra-violet divergent diagrams of the three dimensional theory, and the four dimensional nature of the true underlying theory can and does cutoff the divergences of the three dimensional theory. This has the effect of making all renormalizable terms in the three dimensional gauge theory temperature dependent in a way which may be only computed in the four dimensional theory. After insertion of these temperature dependent masses, the divergence associated with the four dimensional theory are canceled. (For the 1+1 dimensional theory, this only results in logarithmic renormalization of the 1 dimensional theory). In electroweak theory, we find

$$M_Z(T) = M_Z(0)\sqrt{1 - T^2/T_c^2} \tag{180}$$

where

$$T_c = v(0)\sqrt{\frac{1}{1+\frac{3}{8}g^2/\lambda}} \tag{181}$$

In the remainder of the analysis in this lecture,we consider the case $\lambda/g^2 \sim 1$. We have no a priori reason to expect a rapid dependence on this variable, and it is little known. Setting $\lambda/g^2 = 1$ has the advantage that there is no parameter in the theory which is small or large except the three dimensional coupling constant.

Following the analysis of the last section we may now easily estimate the rate for topological charge changing processes. We have and overall factor of $\omega_1 \sim M_Z$ in front of our expression. There is a factor of the volume of the zero mode group, and zero mode normalizations $\overline{NV}$, a factor of $g_3^{N_0}$, and a factor of order one with no dependence on the coupling constant, to leading order when the coupling is small, arising from the contribution to the determinant from non-zero modes. We let this last factor be $\kappa \sim 1$.

We find for the rate therefore

$$\Gamma/V \sim \frac{\omega_-}{2\pi}\overline{N}_{tr}(\overline{NV})_{rot}g_3^{-6}e^{-\beta E_{sp}} \times \kappa \tag{182}$$

$$\sim \frac{\omega_-}{2\pi}\overline{N}_{tr}(\overline{NV})_{rot}\left(\frac{\alpha' T}{4\pi}\right)^3 \alpha_3^{-6}e^{-\beta E_{sp}} \times \kappa \tag{183}$$

From the Appendix, we have for $\lambda = g^2$ that

$$\overline{N}_{tr} \sim 26, \quad (\overline{NV})_{rot} \sim 5.3 \times 10^3 \tag{184}$$

In Fig. 9, the rate of baryon number change is plotted using the result above. Notice that at a temperature of order $T \sim 1Tev$, where the three dimensional coupling constant is small enough $\alpha_3 \sim .1$ so that weak coupling methods should be reliable, we have a rate per particle which is about 10 orders of magnitude faster than the expansions rate. Therefore $B + L$ non-conservation cannot be ignored in electro-weak theory in cosmology.

What happens at temperatures larger than the symmetry restoration temperature of the electroweak theory? At $\alpha_3 = 1$, we see that our result tends to

$$\Gamma/V \sim \alpha'^4 T^4 \sim M_{mag}^4 \tag{185}$$

where M_{mag} is the distance scale over which magnetic interactions are screened in electro-weak theory at temperatures $T \geq T_c$. This result is similar to that for the 1+1 dimensional O(3) sigma model, and is very suggestive. It is also possible to make a guess about the relevant field configurations through which there are transitions, and come to a formula the same as above.[13]. To do this estimate one must integrate over field configurations which are not solutions of the equations of motions, and which also are in a region where semi-classical methods are unreliable. Nevertheless, it is plausible from what we have seen so far that such a formula might be true.

As discussed above, we expect that as $T \to T_c$, the sphaleron becomes of infinite extent, and should decouple in the dilute gas limit. We see that this in fact happens in our expression. As $T \to T_c$, the three dimensional coupling constant α_3 is diverging, and the factor of $\alpha_3^7 \to 0$. This conclusion while consistent with our intuition on the sphaleron solution is probably misleading. In precisely this region, weak coupling methods break down, and although there is probably not a simple classical thermal transition. The dilute gas is probably not dilute. Presumably it is easy to get over the top of the barrier with solutions which do not skim through the saddle point.

4.4 THE PROBLEM WITH INSTANTONS

At first sight, it would appear that there is no contradiction between the estimates using sphalerons and those of instantons for the rate of topological charge change in electroweak theory. The instanton estimate is in Euclidian time, and is used to compute a quantum tunneling probability. The sphaleron can be thought of in real time as used to compute the rate of transitions of particles over the top of a barrier.

The problem arises if we consider the amplitudes which describe baryon number change. Such an amplitude is given generically by $\langle qqql \rangle$ where q is a quark field and l a lepton field. Actually in theories with multiple generations we need three quark fields for each generation and a lepton field for each generation. The amplitude $\langle qqql \rangle$ can be computed in Euclidian space and then analytically continued back to Minkowski space.

In Euclidian space, the only sector of the theory where there is a non-zero zero amplitude for this process is in the winding number one sector of the theory. This can be seen by doing a space-time independent baryon number rotation as in the first lecture. Therefore by the bound we proved on the action in the winding number one sector of the theory (second lecture), we know that

$$| \langle qqql \rangle | \leq e^{-S_{inst}} \tag{186}$$

Therefore the rate must be tiny.

The problem with the above argument is that I believe it is true. It is also irrelevant. The sphaleron is intrinsically a multi-particle process. The sphaleron has a size of order $1/M_Z$, and a typical Fourier component of the sphaleron wave function has momentum M_Z. On the other hand, the sphaleron has energy M_Z/α, and therefore sphaleron processes on the average should involve transitions involving $1/\alpha$ quanta. The typical matrix element which is large, I claim, is $\langle qqqlA^{1/\alpha}\Phi^{1/\alpha}\rangle$.

To see that this is consistent with the sphaleron analysis, suppose the final state distribution of particles is Poisson distributed. The probability of decaying into a state with n particles is therefore

$$P_n = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle} \tag{187}$$

The sum over all n is

$$\sum P_n = 1 \tag{188}$$

This Poisson distribution is consistent with the facts as we know them about the sphaleron and instanton processes. For small n, the rate is $P \sim e^{-\langle n \rangle} \sim e^{-1/\alpha}$, typical of an instanton process. For $n \sim \langle n \rangle$, the rate is of order 1.

For classical decay processes, coherent states, etc., we expect that the decay products are Poisson distributed. This assumption is not too bad. One can question whether the thermal system is sufficiently equilibrated so that a large classical excitation such as the sphaleron is ever excited. Recall however that this is precisely what we have computed when we computed the rate of change of topological charge. The only assumption we made was that the W,Z and Higgs fields were in equilibrium, an assumption which should be very good in cosmology where the expansion rate is slow. We did find a suppression, in addition to the

Boltzmann factor, for making a sphaleron, but this factor involved a few powers of α and was not sufficient to take the sphalerons out of thermal equilibrium at high temperatures.

We can understand the equilibrium argument from another perspective. If the Higgs, W, and Z fields are in equilibrium, then as a consequence of detailed balance the rate of producing sphaleron, R_{prod} is related to the decay rate for the sphaleron as

$$R_{prod} = e^{-\beta E_{sph}} R_{dec} \tag{189}$$

We expect the sphaleron decays in a time of order $1/M_Z$. The rate of production is therefore substantial in cosmology for $M_Z \ll T \ll E_{sph}$, and naively should be large for higher temperatures.

Even if the Higgs, W and Z fields are not in thermal equilibrium, the formation rate can be estimated, and a result similar to the above obtains. This result is at first sight surprising, and one is tempted to say it must be wrong. After all, in a thermal ensemble, nuclear states are not often made even though their Boltzman factor allows it. Several comments are in order about this apparent contradiction. First, the sphaleron is composed of bosons, and their distribution functions are more singular at small momenta, and the sphaleron itself is composed of low momentum quanta. Second, for a nucleus, the typical binding energy is very small compared to the mass of the nucleus. It is very difficult to assemble a nucleus in its ground state because the overlap of thermal distribution functions with the nuclear wavefunction is so small. In the case of the sphaleron, an explicit computation shows this is not the case.[14] Finally, the sphaleron itself is a collective excitation of presumably very many states, and the quantity to compare the probability of making a sphaleron to is the probability of having an assembly of excited nucleons with a typical nucleon momenta of the order of αT, a probability which is not so small as the probability to make a nucleus in its ground state.

The proof of the sphaleron pudding is in its eating however. There have now been a variety of computations of sphaleron induced rates at high temperature, and in contra-distinction from instanton estimates, a large rate results.

5 Appendix A: Sphaleron Zero Modes

In this appendix, I discuss and derive expressions for sphaleron zero modes. I first discuss global gauge rotations of the sphaleron and of the vacuum. At spatial infinity, the sphaleron fields approach

$$\vec{A}_{sp} \to 0 \quad \Phi_{sp} \to \sqrt{2} v \hat{r} \cdot \tau \tag{190}$$

and the vacuum fields are

$$\vec{A}_{vac} \equiv 0 \quad \Phi_{vac} \equiv v/\sqrt{2} \tag{191}$$

Global gauge rotations change Φ in both cases. We fix this gauge degree of freedom by requiring that the fields have the behavior of the above two equations. Any small fluctuations, when combined together with the original sphaleron solution must preserve the behavior above.

First consider translations. We rescale the fields as

$$r \to gvr \quad A \to vA \quad \Phi \to v\Phi \tag{192}$$

so that our rescaled fields and coordinates are dimensionless. With these dimensionless fields, translations give

$$\delta\vec{A} = (\vec{\epsilon} \cdot \vec{\nabla})\vec{A}_{sp} + \vec{D}\Lambda \tag{193}$$

$$\delta\Phi = (\vec{\epsilon} \cdot \vec{\nabla})\Phi_{sp} + i\Lambda\Phi \tag{194}$$

Here the gauge transformation Λ is chosen so that the boundary conditions on the small fluctuations are satisfied,

$$\Lambda = 2\frac{k(\xi)}{\xi}\hat{r} \cdot \vec{\epsilon} \times \tau \tag{195}$$

where

$$k(\xi) \equiv \xi \int_{\xi}^{\infty} d\xi' \, \frac{f(\xi')}{\xi'^2} \tag{196}$$

When the contribution of zero modes to the path integral is evaluated, we must compute factors such as the normalization of the zero mode. For these modes, we find a factor of

$$N = \Pi \left(\frac{1}{2\pi} \int (\delta\Phi)^2 \right)^{\frac{1}{2}} \tag{197}$$

For the translational zero modes we may evaluate the normalization integral. We find

$$N_{tr} = \left\{ \frac{2}{3} \int_0^\infty d\xi \left(\frac{8}{\xi^2}[(f+k-2fk)^2 + (f-k-\xi f')^2] + [h^2(1-k)^2 + \frac{1}{2}(\xi h')^2] \right) \right\}^{\frac{3}{2}} \tag{198}$$

For the case where $\lambda = g^2$, we have $N_{tr} = 26$.

For rotations, $\delta\vec{r} = \vec{\epsilon} \times \vec{r}$, we again must make a gauge rotation to preserve boundary conditions. We find that

$$\delta\vec{A} = 2\frac{1-f}{\xi}[\vec{\epsilon}(\hat{r}\cdot\tau) - 2\hat{r}(\hat{r}\cdot\vec{\epsilon})(\hat{r}\cdot\vec{\tau}) + \vec{\tau}(\hat{r}\cdot\vec{\epsilon})] \tag{199}$$

$$\delta\Phi = 0 \tag{200}$$

We find the normalization integral for rotations to be

$$N_{rot} = \left\{ \frac{32}{3} \int_0^\infty d\xi \, (1-f)^2 \right\}^{\frac{3}{2}} \tag{201}$$

For the case where $\lambda = g^2$, we have $N_{rot}V_{rot} = 5.3 \times 10^3$ We have used that the volume of the rotation group for SU(2) is $8\pi^2$.

Figure Captions

1. Fig 1: The elementary pendulum in a gravitational field

2. Fig 2: The periodic potential for the elementary pendulum.

3. Fig. 3: A half filled Fermi sea for massless particles. Here the eigenstates of $\gamma_5 = \pm 1$ are shown in the two columns.

4. Fig. 4: The result of applying an external field to the fermion system. Here we have made a transition of a state of negative chirality from below the Dirac sea into the continuum, and Dirac sea downward creating a hole in the Dirac sea.

5. Fig. 5: a) The energy levels of the Dirac equation in the absence of the externally applied field. The two axes correspond to positive and negative helicity b) The energy levels in the presence of an external field. Here positive helicity states cross into the continuum and negative helicity states are shifted down forming holes, or anti-particles.

6. Fig. 6: The stationary phase points on a torus. There is a class of non-contractible loops on the torus shown with the dashed line, which passes through the two stationary phase points.

7. Fig. 7: The intersection S_1 between the plane and the two sphere S_2. The points on the plane are parameterized as $p_2 cos\mu - p_4 sin\mu = cos\mu$

8. Fig. 8: A potential with a metastable minimum. The height of the barrier is V_0, and doing a Gaussian approximation around the metastable minimum gives frequency of ω_0, and around the maximum $i\omega_-$

9. Fig 9: The rate of baryon number change in the electro-weak theory as a function of temperature. The rate per particle is about 10 orders of magnitude faster than the expansion rate of the universe when $\alpha_3 \sim .1$. This is at a temperature of $T \sim 1\ Tev$ Here $\lambda/g^2 = 1$

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